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Symbolic representation in mathematics: fMRI-based neuroeducation perspectives on the hypothesis that symbolism is a relief for our brain when thinking about fractions.

Abstract

Eine symbolische Darstellung von Mathematik wird für das Lehren und Lernen häufig als schwierig oder herausfordernd beschrieben. Aktuelle neurowissenschaftliche Erkenntnisse legen jedoch nahe, dass diese für unser Gehirn besonders effizient zu verarbeiten sind. So beschränkt sich die Anzahl aktivierter Hirnareale in symbolischen Zusammenhängen auf eine relativ geringe Anzahl von Hirnarealen und erscheint gleichzeitig als schneller verarbeitbar. In diesem Artikel befassen wir uns mit der Forschungsfrage, inwieweit der Umgang mit symbolischen Darstellungen das Gehirn entlasten kann. Wir gehen dieser Frage mit einem deskriptiven Ansatz und mit Hilfe einer theoriebasierten Diskussion nach und nutzen dabei aktuelle fMRI-Studien, die innovative Erkenntnisse für die Mathematik bereithalten.

Key words

symbolic representation, fractions, decimals, fMRI, brain.

Motivation

The overall goal of this paper – to examine symbolic representations in mathematics education from a neuroscientific perspective – is an important concern for (new) impulses. The aim is bridge between neuroscience and teaching and learning research. Regarding the restriction of the topic to size comparison between common fractions and decimals, a review article with integrated theory-based discussion can provide a link between neuroscience and concepts of mathematics education. This article thereby elaborates the implications of neuroscience research for teaching practice.

As a possibility for the connecting bridge and between the above-mentioned perspectives (cognitive science and neuroscience approaches), we aim to create an explanatory epistemic dimension for mathematics education: an integrative cognitive–neuroscientific approach (cnA) (Pielsticker, 2022).

Let us briefly describe what an integrative cnA is about with an example. Integrative cnA can be described though "practicing" (Pielsticker et al., 2020). The concept of practicing already has a long research tradition in mathematics education. With the help of reconstructions, conclusions are drawn from empirical observations of practicing processes. According to Winter (1984), for example, practicing can provide relief from cognitive burden, and generate new capacity for (demanding) problem-solving processes. For example, series of tasks (addition, according to Selter, 2004) are observed to describe the effects of practicing. A neuroscientific view taking these into account adds a new descriptive access to these findings: cognitive relief through neuroplasticity (structural adaptions). This involves measureable growth in the cerebral white matter (Klein et al., 2019). This relief effect can also be interpreted neuroscientifically as a reduction in the necessary brain areas involved. Ischebeck et al. (2009) described a tendency of the brain to automate. With the help of integrative cnA it is possible not only to conclude from the observed effects that task series have on practicing processes, but also to recognize and describe results from the measurable training effect of the brain through an explanatory epistemic dimension for mathematics education. Combining these perspectives can create a broader basis for argumentation.

Introduction

Symbolic representations play an important role in mathematics, and notably in the successful acquisition of mathematical skills and abilities. Symbolic representation, according to Bruner (1967), is crucial for individual learning processes. In the context of mathematics education, Lambert (2011) makes the following suggestion: that the aim should be to stimulate the ability for a flexible intermodal transfer between the representations (enactive, iconic and symbolic).

In this case, the term 'symbolic' is understood as a property of a sign, a property that is specified by rules for dealing with the sign (Lambert, 2015). However, students seem to develop syntactic rules for a sign contextually (Bauersfeld, 1983). At the same time, dealing with a framing empirical application seems to be important (Pielsticker, 2020). Using the rules developed for a variety of situations, application settings, and linking them often seems difficult. This is also described in an example by Silver. In a study with college students, the task $\frac{1}{4}+\frac{1}{6}$ was solved correctly with the help of fraction bars, but on the symbolic level the answer was $\frac{1}{4}+\frac{1}{6}=\frac{2}{10}$ (Silver, 1986). Here, a college student responds after being confronted with the discrepancy in her score: "Well that's the answer when you are working with bars and other is the answer when you are working with numbers" (Silver, 1986, p. 190). Hefendehl-Hebeker and Schwank consider similar situations when they state that a lot of mental structuring must be applied to representations before the salient information can be taken from them. Their circle of understanding between "reading off" and "reading into" is a problem of its own (Hefendehl-Hebeker & Schwank, 2015). Also, Bauersfeld (1983) formulates a crucial point about reading off and reading into the text. The prerequisite for seeing things in this special way is a definition, a theoretical foundation: a theory is needed to be able to see in a particular way (Bauersfeld, 1983). Challenges thus seem to lie, on the one hand, in the described intermodal transfer between the levels (Lambert, 2011; Radatz, 1991; Moser Opitz, 2007; Leisen, 2010), and, on the other hand, in the transfer of the symbolic level to different empirical applications. The latter also parallels Tall's (2013) "Three Worlds of Mathematics" that describe the transition from "practical mathematics with experiences in shape & space & in arithmetic [to] theoretical mathematics with definition based on known objects and operations" (Tall, 2013, p. 151). In this article, we want to focus especially on the latter. For us in the following, the symbolic representation is always connected to an arithmetic/algebraic expression in the sense of Bruner (1967).

Against the background of an integrative cnA we want to discuss new impulses for the description of symbolic generalizations in mathematical educational research. In some studies of mathematics education research, the symbolic representation is associated with a discharge process. For example, Prediger (2009) speaks of a "denkentlastende Abkürzung [thought-relieving shortcut]" (Prediger, 2009, p. 12) or a "denkentlastenden Kalkül [thought-relieving calculus]" (Prediger, 2009, p. 18). On the other hand, the symbolic level of representation offers a relief from the "burdensome" context of application. Through an integrative cnA we develop further insights for this.

The integrative cnA includes, on the one hand, that the selected publications investigate functionalities of the brain, an in particular that functional magnetic resonance imaging (fMRI) is used for data collection and presentation (compare also the end of this section). With the help of fMRI, brain activity can be measured dynamically, allowing the dynamic flow of thought to be made visible and described. In short, the goal here is to develop (neuroscientific) discussion on mathematical knowledge development processes. Therefore, we would like to focus on an analysis of neuroscientific studies and their results, with concepts from mathematics education. To this end, we will explore the following research question: To what extent is dealing with symbolic representation relieving for the brain? We use the method of theory-based discussion. Thus, we pursue a descriptive approach. "Relieving for the brain" means, here, for example, a higher neurofunctional "internal" rationalization effect which is expressed through a reduction in the brain capacity involved. This means one or both of the following: reduction in the number of brain centers involved, or reduction in the effort for that brain center which is indicated by less oxygen consumption, less blood flow, etc. Including that a symbolic representation can be processed easier and faster than other possible representations. The difference can be understood in terms of a mental rationalization and relief process by which we can arrive at a solution sooner and with less expenditure of energy. What we call the 'internal calculative tendency' of the brain describes how the brain carries out rationalization processes. It reduces the number of activated brain centers in networks, and the develops monocentric focal areas which take care of the whole process while saving oxygen, glucose and perfusory volume. We will illustrate this with an example. When learning to play the piano, one initially needs a high number of brain centers to deal with reading the notes, motor control of the hands, shoulders and arms, higher-level processing, acoustic centers, and auditory comprehension. Through practice, some piano players develop a center that replaces and relieves the main centers involved, and that learns to operate all of the required functions and components. This center is localized in the speech center, and possibly subordinate to it. Such learning thus entails reducing the energy consumption and effort associated with multi-center processing to a center that combines control, coordination, execution, understanding and optimization of the ability. Thus, it is not symbolic representation in itself that provides relief for the brain but the process by which the brain deals (calculatively) with this symbolic representation. This distinction can be easily illustrated by the example of the summation sign. The sign itself is rather complicated to interpret even though dealing with it in complex argumentation reduces calculation efforts substantially.

Opportunities and challenges of using fMRI

Focusing on fMRI brings both opportunities and challenges. The imaging technique offers new possibilities to look non-invasively into the functional processes of the brain. Other medical neuroscience diagnostic tools are also available, such as electrophysiological measurement, magnetoencephalography (MEG), and further imaging techniques such as positron emission tomography (PET). Electroencephalography (EEG) is able to show electrical activity of the brain by recording voltage fluctuations at the surface of the head. Today, it is one of the standard examination methods in neurology. MEG records magnetic, rather than electrical, fields due to the cerebral cortex. It offers higher temporal and spatial resolution than EEG (Bähr & Frotscher, 2009). PET is a nuclear medical diagnostic procedure and examines metabolic processes. Short-lived radioisotopes are injected into the body. A disadvantage is the sometimes considerable radiation exposure and the high examination effort. In addition, the spatial and temporal resolution may not be sufficiently high (Bähr & Frotscher, 2009). The main characteristics of fMRI are the detection and differentiation capabilities of the different magnetic properties of oxygenated and deoxygenated hemoglobin. When a brain area is activated, the blood flow changes, and thus also the oxygenation state. This change can be measured in comparison to the resting state (Bähr & Frotscher, 2009).

An advantage of using fMRI is that a holistic picture can be obtained. It is possible to look at the brain as a whole and gain insight into changing connectivity structures and changing organizational forms of "thinking." In this sense, a "whole picture" is assembled.

Thus, fMRI is particularly well suited to our research question. With the help of fMRI, a differentiated view of the use of (different) symbolic representations can be obtained. This allows a description of different activations of brain areas. Below, we will briefly outline which brain areas will be important in this discussion.

Brain regions with regard to their function

The intraparietal sulcus (IPS) is a special area of the parietal lobule in terms of its functionality and functional subregions. It is located between the lobulus parietalis superior and medius, and consists of a horizontal portion and an oblique portion. Five regions can be distinguished in the intraparietal sulcus: an anterior, lateral, ventral, caudal and medial region. The lateral and ventral regions have the function of visual attention, the ventral and medial region control reaching out to visual targets and pointing, the anterior region controls grasping and modeling hand movements, and the caudal region of the IPS has the function of visual depth perception (stereopsis). Of particular interest is that the IPS plays a major role in the processing of symbolic and numerical information (e.g., in size comparison), as well as in visuospatial working memory. (Fortscher, Kahle & Schmitz, 2018; Kandel et al., 2013). The IPS is also described as being involved in pathological phenomena such as diagnosed dyscalculia. Here, structural and functional changes exist in the IPS and the prefrontal cortex, such as a reduction of gray matter volume in childhood (Szűcs et al., 2013).

The left middle temporal gyrus appears to play a special function in the treatment of fractions. This finding was captured by the study of Cui et al. (2020). This is a new functional discovery that has yet to be placed within the overall functional complex of the lobus temporalis. So far, the following functions of the lobus temporalis are known: hearing, intersection of auditory and visual functions, recognition of spoken and written words (lexical abilities), language functions, ability to analyze acoustic signals, speech production and comprehension, and spatial and visual orientation. The medial and paramedial portions of the lobus temporalis on the left side (in the right-handed person) seem particularly interesting because they contain the hippocampal formation, as well as the parahippocampal and perihippocampal formation. These structures are crucially responsible for declarative memory, for the factual knowledge of semantic memory, as well as for the storage of episodic memory. In particular, the hippocampal and parahippocampal gyri play a major role in the generation of new knowledge content and explicit knowledge. The more basal parts of the temporal cortex host the working memory function, as well as the function of selecting knowledge content for its transfer to long-term memory (long-term memory selection) (Fortscher, Kahle & Schmitz, 2018; Kandel et al., 2013). The authors of this paper place the discovery of the left middle temporal gyrus as an important center for fractional computation into the context of a higher-level function of the perihippocampal formation and, thus, in an intermediate center between working memory and long-term memory. This hypothesis remains to be investigated and tested by further fMRI measurements in the future.

We systematically describe and classify neuroscientific findings, relevant to thinking about fractions (fractions and decimals), in terms of mathematics education. The "Methodology" section describes the method and presentation of the results. But first, we will discuss the concept of symbolic representation and describe our view on it.

Theoretical Approach

A Symbolic Representation

We would like to revisit two observations from Bruner (1974). The first of these is that symbols are arbitrary, and their meaning is variable. The second is that the arbitrariness of symbols allows for regular transformations of sentences that can yield (new) useful statements. Here, the context of a symbol seems crucial. Thus, not only the semantics but also the syntax becomes significant (Lambert, 2011). Lambert (2011) states that symbolic representations (and ideas) contain (recognized) rules and grow beyond themselves through these rules. Isoda (2018) emphasizes the diversity that symbolic representations entail and further incorporates this concept into the approach to mathematization of Nabeshima and Tokita (1957). Arcavi states for his "construct symbol sense" that "symbol sense [...] opens up several issues for reflection, discussion and research" (Arcavi, 2005, p. 43). According to Arcavi (2005), symbol sense is a prerequisite for the (further) development of mathematics. Furthermore, a symbol sense should be built up for the development of mathematical knowledge (Arcavi, 2005). Against the background of considering a historical (further) development of Leibniz's calculus, it is to be noted that this is not least due to Leibniz's talent for a symbolic language that is technically easy to handle (Witzke, 2009). Further, Leibniz's symbolic language continued to implicitly lay the foundation for a rapid extension of the theory to as many areas of application as possible (Witzke, 2009). Additionally, the symbolic–algebraic tool also appears to develop a central importance for the development of mathematical knowledge with respect to Euler (Witzke, 2009). Overall, from the perspective of mathematical educational research, it can be stated that symbolic language seems to have a decisive influence on (further) mathematical development (Arcavi, 2005; Witzke, 2009).

A Symbolic Representation of Fractions

This article is a systematic consideration of neuroscientific findings on examples of fractions (fractions and decimals). Obersteiner and colleagues have already investigated "the cognitive challenges of learning fractions" (Obersteiner et al., 2019, p. 142) and include current results on "neural correlates of fraction processing" (Obersteiner et al., 2019, p. 145). It is interesting that "all existing studies examined the neural correlates of fraction processing in adults, and studies on the neural correlates of how fraction processing develops and shapes the brain are completely lacking" (Obersteiner et al., 2019, p. 148). We hope to be able to contribute to filling this lacuna.

We want to classify the representations of fractions according to the work of Prediger and Wessel (2011). Doing so will allow us to view or name the representations of fractions from the neuroscientific studies in a more differentiated way. We chose the model of Prediger and Wessel (2011) for two reasons: the model allows a detailed look at the individual representations, and at the same time provides the possibility of a networking of the representations. The representations of fractions of this article (the items from the neuroscience studies) can be classified in terms of that model (Prediger & Wessel, 2011). For us, it is interesting to distinguish between the categories of "physically experienceable representation", "graphical representation", "symbolic–numeric representation" and "symbolic–algebraic representation" (Prediger & Wessel, 2011). The representation of fractions that we will consider in this article can be classified into "symbolic–numeric representation" and "graphical representation".

According to Neumann (2000), it is interesting to ask (in mathematics education) whether fractions and decimals are two different ways of writing the same numbers or not. Neumann (2000) investigated whether students believe that fractions and decimals are two completely different kinds of numbers with little or no connection to each other. According to Neumann, 40% of the 129 students studied, with an average age of 13, think that fractions and decimals are two different ways of writing the same numbers. It is interesting to consider whether fractions and decimals are different symbolic representations. As we will see in the neuroscience studies in the "Results" section below, this is an important question with respect to the processing of symbolic representations in the brain. If we look at how fractions are introduced in the classroom, Freudenthal (1983), for example, gives four stages:

- fractions in everyday language. This refers to simple fractions, such as $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}$, etc., which are mentioned in connection with quantities.
- fractions as divisors. Concrete wholes are divided into equal parts. The fraction number indicates the ratio of the fractional part to the whole.
- fractions as comparative ends. Fractions are compared with each other, which are based on the same and also different quantities.

- fractions as operators. Now pure fractions also appear and simple rules are introduced.

To go into further detail, Neumann (1997) distinguishes four aspects of fractions in his study: Part from whole; operator; measure; fractions and measure; decimals. With respect to the aspect "part of the whole" Neumann (1997) can differentiate seven partial aspects: Numerator – number of fractions to be taken, Denominator – number of all fractions of the whole, Equivalence of fractions, Exhaustion of the whole, Fraction is divisible whole again, Relation between the denominator and the number of divisions.

We would like to discuss the differences between a symbolic representation of fractions and decimals. In school classes, the notation of fractions is introduced first, before the students also deal with decimals. The structure of the writing figure (notation) of the symbolic representation of fractions is to be regarded as 'further away' than that of the decimals. If we can find an analogy to natural numbers for dealing with the symbolic representation of decimals (fractions having the same denominators) (Padberg & Wartha, 2017), the writing figure (notation) of fractions, on the other hand, can be classified as novel. Furthermore, the fraction bar, as part of the notation (the 'part of the whole' notion), offers an additional challenge in mathematics education.

In addition to the above classification into different categories of representations, we would also like to return to the domain-specificity of knowledge at this point. Thus, our results from the discussed studies can still be assigned to specific contexts or have emerged in specific contexts. In the following, we will discuss the methodology of theory-based discussion. Based on this, we will describe the studies obtained and discuss the results. As a further aspect for the argumentation and discussion of our research question, we will take a look at polytopic neuropathology in the form of a subvariant of autism (Autism Spectrum Disorder, ASD) and its connection to symbolic representation. ASD can be either a high-functioning form of autism or Asperger syndrome. The idea behind this discussion is that atypical developments can inform our understanding of typical developmental processes. People with ASD often have no choice but to filter and deal with the many individual pieces of mathematical information according to symbolic forms that their brain is capable of processing. On the one hand, this protects them from cognitive overload. On the other hand, the tendency to internal calculation plays a role, since people with ASD only practice these mathematical skills. Non-symbolic content is not cognitively captured in the first place, and thus not (further) processed (Hiniker, Rosenberg-Lee & Menon 2016; Kandel et al., 2013). What is interesting here is how this clinical picture can inform our research question.

Method: Theory-based Discussion

For our results, we selected relevant articles that were conducive to our research question – *To what extent is dealing with symbolic representation relieving for the brain?* For the choice of publications used for our theory- and literature-based discussion, we follow three steps: 1) Planning the research and database research;

2) Conducting the research and selecting the publications; and 3) Systematic discussion of selected publications. The first step mainly determines the motivation of the research. The "conducting the research and selecting the publications" step begins with identifying relevant research findings, and the third step is where the findings are processed and discussed. The database we used was ScienceDirect (https://www.sciencedirect.com/). It covers scientific research on current topics and new findings. Furthermore, this database is characterized by a wealth of current scientific publications in health care, and it serves especially frequently for cross-disciplinary research. We have included studies published from 2015, as technology in the medical field – such as the high-tech process of magnetic resonance imaging – is evolving rapidly. The first step in this process was the database search, for which we used the following search string: "fraction learn* fmri" in titles and abstracts. This initial search identified 678 matches. The second step, winnowing the selection, sorted through the titles, abstracts, and the conclusions for studies meeting the following criteria: the study was published in a journal, the study was written in English (in the sense of accessibility), the topic was substantially related to the teaching and learning of fractions in mathematics. Winnowing left eight studies. In the third step, these studies were read in their entirety and reviewed in full for stricter winnowing, leaving three recent studies that meet our criteria and are conducive to our research question. Thus, the theory- and literature-based discussion in our study has been very narrowly framed. There are several reasons for this. First of all, we focus on fractions as there is a long tradition of studying this content area in mathematics education. This helps to connect our study to well-known grounds and makes it accessible for discussion. As we have argued in detail above, we are now very much interested in the results of fMRI studies as they provide us with new insights on processes of the brain when dealing with fractions. This, as we see, of course limits the number of studies substantially, which is of course a research finding itself. Nevertheless, from our point of view, it is possible to give a first impression of the potential contribution to an integrated cnA.

Table 1 lists the studies that we want to discuss. It is striking that in all three studies, magnitude and proportion processing and part–whole relations were the focus of the investigations in addition to the reference to fraction representations.

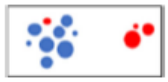
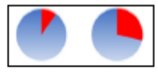
Studies	Journal	Aim of the study	Findings	Possible items
Mock et al. (2019)	Brain Research	The aim of the study was to investigate whether there is a common neural network for processing magnitudes and proportion in the intraparietal cortex. "[…] the present study aimed at investigating whether there is a shared neural correlate for proportion processing in the intraparietal cortex beyond overall magnitude processing and how part-whole relations are processed on the neural level" (Mock et al., 2019, p.133).	In particular, the intraparietal sulcus (in short: IPS) was detected as the main center in all forms of representation. It apparently plays a decisive role in the processing of proportion and magnitude comparison task. Furthermore, the IPS also seems to be important in processing part-whole relations of symbolic and non-symbolic representations.	$\frac{8}{9}$ $\frac{2}{5}$  0,89 0,40 
DeWolf et al. (2016)	Neuro Image	The aim of the current study was to answer the question of how different symbolic representations (natural numbers, fractions and decimals) are neuronally mapped.	The authors were able to show that magnitude comparison tasks related to natural numbers and decimals are located in a specific area in the IPS region (especially on the left side among right-handers). In contrast, magnitude comparison tasks involving fractions show activation in the bilateral IPS.	Fraction pairs: $\frac{1}{9}$ vs. $\frac{3}{7}$ Decimal pairs: 0,11 vs. 0,43 Integer pairs: 11 vs. 43
Cui et al. (2020)	Neuro-science Letters	Cui, Li, Li, Siegler and Zhou (2020) conducted a study looking at functional activity in the left middle temporal gyrus.	The authors conclude that the left middle temporal gyrus is a key region of the brain that critically influences magnitude comparison tasks involving fractions.	Fractions: $\frac{1}{2}$ vs. $\frac{6}{7}$ Whole numbers: 89 vs. 78

Table 1: Key Data of the Studies to be discussed.

We now summarize the results of our listed studies, and discuss them considering our research question. For the description of the individual study results within our theory- and literature-based discussion, we proceed according to the following scheme:

- 1. Explanation of the study question (What was investigated?)
- 2. Description of the subjects (Who was studied?)
- 3. Explanation of the method, material and tasks (How was the research question investigated?)
- 4. Display of results

Results

Regarding the study by Mock et al. (2019)

- 1. Explanation of the study question:

The aim of the study was to investigate whether there is a common neural network for processing magnitudes and proportion in the intraparietal cortex. It was noticed that dot patterns, pie charts, and fractions activated an extended frontoparietal network. This indicates a large overlap of brain areas in large parts. Furthermore, this network shows an activation of up to 29 brain areas for the dot patterns. Of these 29 brain areas, 18 brain areas, in turn, show a striking and significant increase in activity during the tasks posed. A large congruence of the areas was found.

- 2. Description of the subjects:

24 students (right-handed, 13 female, average age 23.2 years) took part.

- 3. Explanation of the method, material and tasks:

The experimental tasks involve comparing magnitudes in different representations. Depending on the specific context of the task, certain knowledge must be activated by the participants (see Tab. 1 for comparison tasks in different contexts between fractions, decimals, dot patterns, and pie charts). Furthermore, the authors of the study note that the dot patterns and pie charts were colored in a certain way. Our Table 1 shows, for a dot pattern example, eight out of nine dots colored in blue. Here, the dots have different sizes so that participants "cannot rely on the proportion of the sum of visual areas when comparing the dot patterns" (Mock et al., 2019, p. 142). In the pie charts, the circles were divided into circle segments according to the sizes. The respective segments, for example, eight of nine segments in Table 1, were colored blue. The overall size of the pie charts was also adjusted in a similar way to the dot patterns. In contrast, 10 brain areas were activated in the network for the decimal numbers, 5 of them significantly. These brain areas were, for the most part, different from the brain areas of the other three forms of representation. In all forms of representation, the main center detected was the IPS, which thus seems to play a decisive role in the processing of proportion and magnitude comparison task. Furthermore, the IPS also seems to be important in processing part–whole relations of symbolic and non-symbolic representations. In addition, the IPS and the adjacent parietal cortex are part of the higher-order association cortex, which is also referred to as the tertiary cortex. In addition to number-processing, sensorimotor–cognitive processes such as attentional orientation, hand orientation and grasping, spatial working memory, planning of saccade movements, mental rotation and navigation are also located here.

- 4. Display of results:

The authors of the study interpret the results as follows: Fractions, dot patterns, and pie charts exhibit a bipartite structure of these presentation formats that activates multiple neural processing functions. Therefore, regarding bipartiteness, the authors claim that: "Fractions, dot patterns, and pie charts reflect bipartite part–whole relations. In particular, the bipartite structure of these presentation formats requires relating their two parts to infer the respective relative magnitude information" (Mock et al., 2019, p. 135). A bipartite part–whole relation of the fractions, dot patterns and pie charts can therefore be understood as forming two parts (bi-parts). Here especially, visual processes, the transformation of a fraction e.g., into a circle diagram and vice versa, and top-down processes etc., are in the foreground. In the case of decimals, these processes do not occur altogether, since a one-partness is given and decimals "[...] simply and directly reflect number magnitudes in a base-10 notation [...] and do not require the additional processing of proportional aspects" (Mock et al., 2019, p.140). It is astonishing that the test items with decimals could be solved with the highest speed and the highest accuracy, compared to the other forms of representation. "In fact, participants responded more accurate and faster to decimals" (Mock et al., 2019, p. 141). "Thus, the structure of proportions (part–whole vs. base-10) seems to determine their underlying neural processing" (Mock et al., 2019, p. 140). We can also illustrate this as follows. When a decimal number is read, each digit specifies the number (it is about). For example, with the number "1,[]" it is clear that this number must lie between "1" and "2" on the number line, with the number "1,2[]" it is clear that it must lie between "1,2" and "1,3" on the number line, and the number "1,25[]" lies between "1,25" and "1,26", and so on. For common fractions, the numerator and denominator of a number must be related. For example, for 12/234, first the first digit "1", of the numerator is read, then the "12", but this alone cannot decide where this number lies on the number line. Then we add the first digit of the denominator, the "2", and here also no classification on the number line is possible; this can be achieved only with the whole denominator and the whole numerator. Numerator and denominator must be put in relation, so that a classification on the number line becomes possible. Furthermore, this can be explained as follows. The network focuses on only a few crucial brain areas that had to be activated. These seem to be further characterized by a mental optimization process (possibly also genetically determined). Thus, the human brain, in the case of magnitudes and proportions, seems to prefer the transfer to a symbolic, unambiguous, non-multipartite form of representation compared to other forms of representation.

Regarding the contribution of the study to our research question: *To what extent is dealing with symbolic representation relieving for the brain?*, the advantages of the symbolic representation of mathematics using the example of decimals can be described as follows: Avoidance of secondary influences from higher-level centers that may expand the cognitive spectrum but manipulate and interfere with the main mental procedure. Meanwhile, the literature views the area of the IPS as a domainspanning multi-demand system (Humphreys & Lambon Ralph, 2015; Humphreys & Lambon Ralph, 2017; Simon et al., 2002). This is attributed to being able to control, in a superordinate manner, centers frontal to occipital as well as in the basal ganglia. The decimals appear as a one-part, self-contained system that triggers very specific attentional processes. Therefore, only the main areas involved in the mathematical calculation process are activated. In the case of looking at magnitudes of fractions, a two-part system seems to appear in the form of representation. This must be processed partly in mathematical preliminary steps, which can trigger conceptions that are linked associatively with what has already been learned, e.g., the handling of pie charts. "This might indicate higher attentional loads as well as working memory demands during additional computational steps to access overall magnitude information of fractions" (Mock et al., 2019 cited after Mock et al., 2018, p. 140). Regarding our research question, it would therefore be obvious to state that, in the context of magnitude processes, the more uncomplicated symbolic mathematical representations appear to the brain, the faster, more correct, and less time-consuming possible solutions become. This "appearance" can be seen in the dependence of already existing (individual) Domains of Subjective Experience (DSE) (Bauersfeld, 1983; 1988) and built-up concepts (e.g., number sense according to Dehaene (1999); or symbol sense according to Arcavi (2005)). Thus, it is not a matter of making a blanket statement that decimals are "easier" for the brain to process, but that we are, in a sense, also used to it. We are used to seeing decimals as a whole, and to seeing common fractions as two parts. This can be illustrated, for example, by looking back into the historical number system of the Egyptians (from about 3000 BC). But the brain's faster processing does not seem to be a phenomenon inherent to the symbols. Rather, this seems to be due to our perception of the symbols and the already built-up DSE (Bauersfeld, 1983; 1988). In the sense of an analogy formation, the brain processes decimals obviously in a similar way as knowledge that has already been acquired (natural numbers). Fractions, however, have a different location in terms of brain processing (see the results of Cui et al., 2020), and also depend on the context (domain-specificity of knowledge, Bauersfeld, 1988) in which prior knowledge was built up and to what extent the symbolic representation is a closer one for brain processing.

Regarding the study by DeWolf et al. (2016)

- 1. Explanation of the study question:

The following study also discusses the presentation of numbers (natural number (positive integers) and rational numbers (fractions and decimals)) (DeWolf, 2016). The present study discusses the extent to which the representation of numbers is crucial for the brain areas addressed when solving quantitative comparison task.

- 2. Description of the subjects:

Sixteen participants from the University of California and Los Angeles (UCLA) participated in the study (12 of whom were female). On average, the participants were 21 years old and there was no evidence of neurological disorders. Participants were made aware of the study through a flyer.

- 3. Explanation of the method, material and tasks:

DeWolf et al. (2016) demonstrated that quantitative comparison tasks (e.g., Fraction pairs: $\frac{1}{9}$ vs. $\frac{3}{7}$; $\frac{1}{2}$ vs. $\frac{3}{4}$ Decimal pairs: 0,11 vs. 0,43; 0,5 vs. 0,75, Integer pairs: 11 vs. 43; 50 vs. 75, (DeWolf, 2016, p. 304)) related to natural numbers and decimals are located in a specific area in the IPS region, particularly on the left side (right-handed). In contrast, quantitative comparison tasks in fraction representation show bilateral intraparietal sulcus activation. Thus, the representation of the tasks in decimal, integer, and fraction representations was crucial for the neuronal activity triggered. The study attempted to differentiate with respect to its question whether quantitative comparison tasks for all three symbolic representations (naturals, decimals, and fractions) were processed at the same anatomical location in the IPS. Or whether fractions and decimals are processed at the same location and (positive) integers at an independent location. Or whether fractions are processed at an independent location and decimals and (positive) integers are processed at a different but same location in the IPS. Surprisingly, the latter was the case.

- 4. Display of results:

The authors deduce that the observed result is due to the symbolic form of representation where computational processes take place neuroanatomically–functionally, and to which network is activated in the conjunction of the brain areas. There are hence different symbolic forms of representation. Decisive for this interpretation is not the content or the degree of difficulty of the quantitative comparison tasks, but the non-partite (positive integers, decimal numbers) vs. bipartite (common fractions) representation. “Thus base-10 notations evoke similar activation patterns based on their relative magnitudes, whereas the bipartite fraction notation is processed very differently from either” (DeWolf, 2016, p. 311). That is, the neural code for number representation for the non-partite numbers (positive integers and decimals) is on the same neurofunctional unit. However, the bipartite representations (in quantitative comparison tasks) are processed differently, or at a different location in the IPS. This is due decimals such as 0.4 and an integer, namely the 100-fold, i.e., the number 40, being neurofunctionally close to each other, whereas common fractions are perceived and processed differently simply because of their bipartite structure.

With regard to our research question, *to what extent is dealing with symbolic representation relieving for the brain?* we note: For decimals and (positive) integers, mainly the left-lateral IPS is used, as well as a dedicated center in the IPS. In contrast, for fraction representations, the bilateral IPS must be activated, as well as the corresponding considerably larger networks. That is, the more simplified and closed the symbolic representation form (and the associated mathematical rules and laws) appear to the brain, the higher the neurofunctional "internal" rationalization effect. It seems as if our brain grasps and processes decimals as a total expression. Due to the already developed DSE (Bauersfeld, 1983; 1988) and the (familiar) contexts, the symbolic representation of decimals – as a symbolic unit (non-partite) – can be processed easier and faster. In contrast to the grasping of a unit (non-partite) it is probably more strenuous for our brain to process single parts (e.g., bipartite structures like common fractions). Thus, decimals and common fractions can be paradigmatic examples of the contexts in which our brain processes symbolic representations (in quantitative comparison tasks). This finding is particularly interesting because teachers often report that students find it easier to deal with decimals than with common fractions (Padberg & Wartha, 2017). Padberg and Wartha (2017) have collected arguments from parents, children, and even teachers against dealing with fractions. If we now consider that decimals are neurofunctionally processed in a similar way as (positive) integers (and thus presumably also as natural numbers, a number space that children usually start with) this issue appears in a new light. Decimals appear to be neurofunctionally more "familiar" in quantitative comparison tasks and thus to provide the students with a more "familiar" access.

Regarding the study by Cui et al. (2020)

- 1. Explanation of the study question:

Cui et al. (2020) conducted a study focusing on functional activity in the left middle temporal gyrus. The authors conclude that it contains a key region which exercises decisive influence on quantitative comparison tasks involving common fractions. This anatomical site also represents a major functional center with respect to the coordination of mathematics-related semantic processing. Zimbardo and Gerrig (1996) define semantic memory as follows: It refers to the symbolically represented knowledge that a person has about the world. It is that part of declarative memory that contains the basic meanings of concepts and words (Zimbardo & Gerrig, 1996).

- 2. Description of the subjects:

In the study, students (average age 21.53 years, 34 female and 34 male participants) were presented with quantitative comparison tasks.

- 3. Explanation of the method, material and tasks:

Participants performed quantitative comparison tasks for common fractions (e.g., $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{6}{7}$) and for natural numbers (10–99). A distinction was made between quantitative comparison tasks on common fractions with a large gap. "The difference between the two numbers was greater than $\frac{1}{2}$ [and with a small gap] the difference was less than $\frac{1}{4}$" (Cui et al., 2020, p.3). The natural number tasks also distinguished between a large difference "greater than 20 [and no difference] less than 10" (Cui, 2020, p. 3).

- 4. Display of results:

Cui and her colleagues (2020) conclude that, in the cognitive processing of fractions, we need to retrieve semantic knowledge in order to extract certain mathematical rules that are necessary for dealing with the given fraction (again, this makes it very clear that context and built-up DSE (Bauersfeld, 1982; 1988) are important). Thus, the study was able to significantly add a major center to the understanding of neural mechanisms underlying number processing – a main center where visuospatial and verbal knowledge contents are encoded. These kinds of knowledge content enable the processing and comparison of fractions, while the number processing of natural numbers is located and focused in the IPS. Here, it needs to be understood that the semantic memory is the memory of the general education, of the so-called world knowledge. This is decisively shaped by knowledge that has been acquired from different subject areas or genres. This knowledge is stored and filed in different brain areas. Furthermore, this knowledge is reactivated and re-coordinated by the superior center, here e.g., the left middle temporal gyrus. Crucial for the function is the knowledge store and the degree of knowledge of the individual general education storing centers, as well as the connectivity and the activation ability of the left middle temporal gyrus. Conversely, this means that a high level of effort must be expended to perform fractional arithmetic. In contrast, decimal arithmetic requires only a few brain areas in the context of quantitative comparison tasks, with the main center in the IPS as the superior site. This may also provide clues as to why students face some challenges in being introduced to fractions, and learning fractions, in the classroom. A high general education level with many saturated general education memory centers that have high connectivity with each other is required. In addition, with the higher-level coordinative center of the left middle temporal gyrus, visual-spatial and verbal encoding, the brain must have a high level of maturation and knowledge to process fractions.

Regarding our question, *to what extent is dealing with symbolic representation relieving for the brain?* we refer to Cui et al. (2020): it is not the symbolic representation per se that triggers a mental rationalization and relief process. Here, a differentiation has to be made. Especially with regard to the complexity of the (symbolic) representation form (DeWolf et al., 2016). Of course, a quantitative comparison of common fractions can also belong to the symbolic representation but seems to exhibit multipartness in the visual–spatial representation form (DeWolf et al., 2016). This appears to result in the immediate displacement of this process to another higher-level functional anatomical center after visual awareness.

This represents a difference from other symbolic representations (e.g., decimals). Thus, quantitative comparison with fractions is processed in the main semantic memory coordination center, whereas the other symbolic representational forms of quantitative comparisons are processed in the left IPS. Thus, one-part and, to some extent, "simplified" symbolism with respect to forms of representation seems to lead to optimally rationalized handling of mathematical processes. Such a one-part symbolism appears to be "more simple" to the brain. In the following section, we will take a look at polytopic neuropathology in the form of a subvariant of autism (ASD) for further argumentation and discussion of our question. In doing so, we will look at how this clinical picture can inform our research question.

Subvariants of autism (ASD) and its connection to symbolic representation

In the following, a sub-variant of autism (ASD) is considered. This is attributed either to an Asperger syndrome or to a high-functioning autism, in which islets of ability or polyvalent high talents can occur. This pathology shows, in an eye-catching way, how the degradation and the associated malfunction of brain areas and networks, genetically determined, leads to a multifactorial giftedness, often in mathematics, language or art. Non-symbolic mathematical abilities thereby decrease in a subtotal way, and thus remain insufficient. On the other hand, symbolic mathematical processing processes in particular are developed and used in an extraordinary way. This is also particularly interesting against the background of an inner rationalization. Thus, we believe that the following consideration of autism can contribute to our research question. The reason for this is that, due to polytopic pathology and negative influence, a brain built up in such a way naturally, compensatory, searches for optimized and internally calculated ways (in some areas also to function overoptimally). And here it is the way of symbolism which is quite clearly pursued (see e.g., study of Hiniker et al. (2016)), i.e. a brain working in this way shifts its attention to symbolism. Thus, the study by Hiniker et al. (2016) provides a crucial impetus for this article. Hiniker and colleagues (2016) conducted a comparative study between 36 children with ASD (32 male) and 61 typically developing children (TD) (54 male), aged 7–12. Participants were asked to solve two types of quantitative comparison tasks: symbolic quantitative comparison tasks (representation with Arabic numerals, e.g., 6 vs. 8), and non-symbolic quantitative comparison tasks (representation with dot patterns). The authors conclude that their "findings suggest that symbolic systems may help children with ASD organize imprecise information" (Hiniker et al., 2016, p. 1268). They note that their research "indicates that symbolic number sense ability holds significantly more explanatory power, over and above non-symbolic number sense, in the ASD group than for their TD peers" (Hiniker et al., 2016, p. 1278).

Neurofunctional and neuropathological changes in autism (ASD)

The term, 'autism' was coined by the Swiss psychiatrist Eugen Bleuler. It was intended to establish a symptom description as a medical term in the context of research into the underlying disease of schizophrenia. The term was intended to describe detachment from reality with predominant perception of a compartmentalized internal life in the affected patient (Bleuler, 2014). Freud expanded this definition to include the broader concept of self-centeredness (Schröter, 2012). Asperger (1944) and Kanner (1943) recognized that autism cannot be a single symptom of an underlying schizophrenic disorder, but a complete neuro-psychiatric disorder of its own pathological entity of varying degrees (Kandel et al., 2013). Today, we speak of an autism spectrum disorder, which is an umbrella term that summarizes the different gradations and previously detected autistic clinical pictures. Approximately 1% of the population is affected by autism and its associated changes in varying degrees (Kandel et al., 2013). In addition to many possible and still controversial causes for the development of autism, genetic factors are considered to be the main cause. More than 100 genes and more than 40 gene loci have been detected that are pathologically altered in different recombinations and are responsible for the neurological deficits (Kandel et al., 2013). In terms of symptomatology, disorders arise primarily in the areas of social interaction, communication, and repetitive and stereotypic behaviors (Kandel et al., 2013). In the context of mathematics education – especially concerning symbolic representations – a subvariant of autism, the Asperger syndrome, is particularly interesting for us. Besides the partly similar negative symptoms, like classical autism, Asperger syndrome offers a number of positive neuropathies regarding phenotypic expression and associated intellectual abilities. "One of the most fascinating features of autism is the existence of so-called 'islets of ability', in at least 10% of the cases, in music, art, calculation, or memory" (Kandel et al., 2013, p. 1675). There are countless reports of above-average intelligence or even of islets of ability. These patients are particularly likely to have islets of ability with respect to mathematical abilities. Different brain regions were thought to be characteristic in the context of autism disorder or were found to show hypoplasia (e.g., the vermis cerebellaris region, or structures in the temporal lobe) (Kandel et al., 2013). We will not go into this in detail in this article.

Asperger syndrome as a form of autism

Asperger syndrome represents a subvariant of autism. "The label 'Asperger syndrome' is often used for individuals who exhibit the typical features of autism but have high verbal ability and no delay in language acquisition" (Kandel, 2013, p. 1664). As a rule, there is often no disorder in the area of language development, nor a reduction in intelligence. Impairments are present in the area of gestures, facial expressions, eye contact and in the area of social interaction and communication. In addition, there are disturbances in the relationship to peers, empathy, joy, interest, or pride in one's own achievements or in the achievements of others. There is a lack of emotional reciprocity. As with other autism of varying degrees, stereotyped patterns of interest are present, adherence to routines and rituals, stereotypes and repetitive motor habits, and especially unusual persistence in engagement with object parts. Unlike conventional autism, Asperger syndrome does not present deficits in cognition, practical skills, and adaptive behavior, except for social interaction. In approximately 10% of cases, Asperger syndrome co-occurs with an intellectual giftedness or islets of ability (Kandel et al., 2013). Statements such as rhetorical questions, figurative phrases (metaphors), figurative expressions, and irony are taken literally and are not understood in terms of meaningfulness (Kandel et al., 2013). Islets of ability can be congenital, based on a genetic disorder or on brain damage. In most cases, the extraordinary ability is based on a missing filter function, so that regarding a completely defined field of knowledge, all contents can enter the brain for processing and must be processed. Current neurophysiological analyses assume that the phenomenon of islets of ability is based on a compensatory mechanism, especially in autistic persons who, in the case of insufficiency of other neurofunctional systems, rely on the contents they are supposed to be able to handle well, in a compensatory way (Kandel et al., 2013). Work by Fitzgerald (2004) considers the cause of the extraordinary abilities in the autistic person to be in the context of neural dysfunction. After conducting multiple fMRI studies with autistic individuals, Snyder (2009) concluded that the ability to turn off certain brain areas gives way to the reserves of other brain areas that might be needed (regarding the savant effect). "Savants access or read off something that exists in all of our brains but is normally inaccessible through introspection" (Snyder & Mitchell, 1999 cited in Snyder, 2009, p. 1399).

Asperger syndrome and the relation to mathematical processes

Mathematically highly gifted Asperger patients usually show an exceptional talent, often an isolated focus on numbers, and especially on calculating with these numbers, the corresponding rules, and also in terms of retaining results even with large sets of numbers and over the course of many years. Moreover, some have the ability not only to solve mathematical calculations extraordinarily quickly and to reproduce the results, for example, to 100 decimal places, but also to adapt their calculation algorithms to new mathematical situations and thus to derive, invent, and apply mathematical rules anew (Kandel, 2013). What is crucial here is that it is a defined and delimitable (mathematical) domain in which an Asperger patient can operate. E.g., when it comes to the domain of natural numbers, which are defined via the Peano axioms. The Peano axioms are decisive for the Asperger patient and these are strictly worked with. An Asperger patient would not look for an application to the established rules (axiom system), e.g., a description of the Peano axioms on the basis – let's say – of a domino row, because he or she then thinks of dominoes, but no longer of the mathematical background. This is supported by a study by Hiniker, Rosenberg-Lee, and Menon (2016). Here, the following finding was made for children with ASD: "We found significant impairment in non-symbolic acuity in children with ASD but symbolic acuity was intact. [...] Numerical acuity was assessed using symbolic (Arabic numerals) as well as non-symbolic (dot array) formats" (Hiniker, Rosenberg-Lee & Menon, 2016, p. 1268). In summary, the study described "that symbolic systems may help children with ASD organize imprecise information" (Hiniker, Rosenberg-Lee & Menon, 2016, p. 1268). This finding on symbolic and non-symbolic number sense in children with ASD is supported also by earlier studies (Meaux, 2014; Mitchell & Popar, 2004; Rinehart et al., 2000; Scripture, 1981; Titeca et al., 2014; Turi et al., 2015). Iuculano et al. (2014), in their fMRI study, find that autistic individuals with giftedness or the predominantly present islets of ability for mathematics activate the same brain areas in the fronto-temporo-parietal network as typically developing controls. In addition, however, analytical centers in the temporo-parietal-occipital cortex are activated that did not contribute to a mathematical solution in the group of typically developing controls. The autistic children with islets of ability or polyvalent giftedness were significantly more numerous than the group of typically developing controls in terms of accuracy and speed in the achievement tests.

Final Discussion

Various works suggest that symbolic representations are crucial for the (further) development of mathematical knowledge (Witzke, 2009; Arcavi, 2005) among other things because they offer the possibility to transfer a theory efficiently to as many application areas as possible (Witzke, 2009). Regarding empirical (mathematical) theories, symbolic representations can be of crucial importance, e.g., in the use of theoretical concepts for which it is in the nature of things that there is precisely no empirical reference (Burscheid & Struve, 2020) (an example is the concept of force in Newtonian mechanics (Burscheid & Struve, 2018)). In fact, it seems downright necessary to formulate theoretical concepts in a symbolic way. Against the background of the integrative cnA and the studies considered in this paper regarding quantitative comparison tasks, it was shown that, with non-partite symbolism (e.g., with decimals), a rationalization effect occurred with respect to the number of activated brain areas and with respect to their degree of activation. On the other hand, the superior centers showed the same degree of activation as with non-symbolic and also multi-partite forms of representation (e.g., fractions, dot and pie charts). Furthermore, it was displayed that symbolic forms of representation, with respect to decimals, brought about a significant increase in efficiency with respect to the solving behavior for quantitative comparison tasks, concerning speed and accuracy (Mock et al., 2019). Quantitative comparison tasks involving decimals are processed in a dedicated area of the left IPS (right-handed) (with a small downstream network) (DeWolf et al., 2016). In contrast, quantitative comparison of fractions (and representations of fractions) takes place in the left medial temporal gyrus (right-handed) (Cui et al., 2020) (provided with large downstream network connectivity). This can be interpreted as high mental effort in fraction arithmetic. In contrast, the effort required for decimal computation and dealing with the natural numbers in the IPS (in terms of magnitude comparison tasks) appears to be much lower. DeWolf et al. (2016) and Cui et al. (2020) attribute the outsourcing of fraction computation to the medial temporal gyrus to the multipartite representation. This can be interpreted as showing that the brain does not conceive of fraction arithmetic as a process processed by the higher-level mathematics center IPS at all, but rather as a process that must be outsourced to another specialized (tertiary) center. As the examined studies focused on symbolic representations in the context of quantitative comparison tasks, they produced only the observation that decimals, in their one-partite depiction, appear to be "easier" on the brain. Nevertheless, we do not think that this insight should be generalized – in contrast, we want to suggest that each (symbolic) application context does favors not only certain symbolic representations, but whole symbolic languages (including "vocabulary" and "grammar"). At this point, we want to formulate that probably individuals who are capable of selecting the most elegant symbolic representation, viz. a symbolic language, for a certain application are the most successful problem solvers in symbolic contexts. This hypothesis invites for further studies using fMRI technology, which will enable us to observe, for example, if, in fraction arithmetic, it might be the fractions rather than the decimals which are processed more effectively by the brain, i.e. activating fewer brain areas. We may speculatively anticipate this due to the flexibility of their depiction in the sense of equivalency classes ($\frac{1}{9}=\frac{2}{18}=\frac{3}{27}=\cdots=\cdots$) and the possibility of representing infinite decimals as fractions of two single digits (e.g. $\frac{1}{9}=0,\overline{1}$). This center coordinates and structures the semantic memory in particular, which apparently must activate many tertiary centers (and thus access more general non-specific knowledge, depending on the level of education), in order to be able to process and solve quantitative comparison tasks in fraction arithmetic. This shows that two symbolic forms of representation, e.g., decimals and fractions, can be received and processed by the brain in completely different ways. The symbolic form of representation does not automatically imply an inner calculation and rationalization of mental processes. Decisive in this context is the kind of representation (single-part or multi-part) in relation to the usage context. Thus, it is obvious that a symbolism appearing as simple as possible to the brain (in a system with fixed mathematical rules) yields the highest inner calculative tendency relative to the given task. This suggests that mathematical knowledge can be built up effectively if not too many sensorily experienced single details have to be processed at the same time. A (more general) reduced processing in this sense seems to be easier to perform for our brain. Effective symbolic representations undoubtedly open a way to this. A look at savants, leads us to the following findings: based on polytopic brain damage and impaired networks, on the basis of impaired blocking and filtering mechanisms, a mathematical giftedness results which is limited to the ability to move in symbolic representations with clear boundaries and rules and to reach extraordinary peak performances here. The ability to deal with non-symbolic mathematical forms of representation, on the other hand, is only rudimentary or not present at all. These patients take in a vast amount of (empirical) mathematics every day without restraint, which would lead to an overload of the brain areas involved, since natural ordering and categorizing mechanisms for the flood of information are missing. They have no choice but to filter and deal with the many individual pieces of mathematical information according to symbolic forms that their brain is capable of processing. With this procedure, the brain protects the organism from a cognitive overload. On the other hand, it is exactly this tendency toward internal calculation that fosters high performance on the symbolic level, as savants intensively and exclusively practice these mathematical skills. Non-symbolic content is not cognitively captured in the first place and thus not (further) processed (Hiniker, Rosenberg-Lee & Menon 2016; Kandel et al., 2013). For a brain trained in this way, empirical ambiguity (Steinbring, 2009) or the semantic location of mathematical knowledge is an almost insurmountable challenge. In the case of fractions of the type ($\frac{1}{5}$), from an integrative cnA point of view, a challenge seems to lie in the fact that these are detected by the brain differently than, for example, decimals of the type (0,5). They appear to the brain as ambiguous in a certain sense. At the level of symbolic–numeric representation (Prediger & Wessel, 2011) for processing in our brain, a more differentiated view and a consideration of domain-specificity of knowledge (Bauersfeld, 1988) seems necessary. E.g., from the perspective of mathematics education, it is remarkable this, in German schools, fractions are often introduced first, before the decimals. In the mathematics curricula of some other countries, decimals are presented before fractions. That this can also be arranged differently – perhaps more appropriately to the neurological and educational findings, at first sight – is shown historically by the examples of Austria and France. In Austria, a connection through fractions and decimals was stimulated by a parallel treatment (Padberg, 1989). In France, there was an emphasis on decimal breaks for a longer period, but there seems to have been a change for some time now as well (Padberg, 1989). This suggests that decimal representations of rational numbers seem to be more accessible than fractional representations only with respect to the specific context of learning. In the final analysis, the use of decimals, like the use of fractions, is subject to certain syntactic rules at the symbolic level. With Wittgenstein (1984), one could also speak of two different Zeichenspiele, which certainly do not differ fundamentally with respect to their degree of difficulty. Many numbers – just think of repeating, non-terminal decimals such as $0,\overline{3}$, for example – can be represented with much less symbolical effort as a common fraction, $\frac{1}{3}$. And this is a fraction that children are certainly already familiar with from different size ranges. What is perceived as easier or more challenging certainly depends partially on the extent to which the syntactic rules show analogies to the already developed prior knowledge. And here, decimals – for quantitative comparison tasks – seem to be more accessible for children who have learned to deal with place values in detail in arithmetic lessons in elementary school. Fractions, on the other hand, although known from many factual situations in everyday life, are more likely to be perceived as being composed of the familiar natural numbers, and equating when adding fractions is seen as counterintuitive compared to adding natural numbers. To support our point, we could look at the following example: What is $\frac{1}{2}+\frac{1}{3}$? The calculation, in fraction representation, seems easier to process ($\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$) than with decimals and periodic, non-terminating decimals like: $0{,}5+0{,}\overline{3}$. For this, the algorithm of written addition would be used, which is not possible at all. (Analogous, this can be thought for multiplication). Regarding our research question, we conclude that symbolic representations (in terms of quantitatively comparing fractions and decimals) show considerably less brain activity in terms of processing speed. For our brain, decimals appear to be easier and faster to process than fractions (or also, for example, pie charts) (with regard to prior knowledge or arithmetic socialization in primary school lessons). The studies discussed here suggest that the representational form of the symbolism (e.g., non-partite or multi-partite) can have a decisive influence at this point (e.g., reduced load, speed and accuracy with fractions). All in all, the symbolic language appears to be relieving for our brain because we arrive at a solution faster and with less expenditure of energy. Calculative processing is cognitively much less challenging than semantically embedded problem solving. With standardized routines, symbolic relationships (e.g., algorithms) can be grasped and processed particularly quickly and efficiently. In the sense of our brain, we are particularly effective if we can follow a formal calculus (once consolidated). This is conceived as operating in the medium of sign expressions according to certain rules for the "correct" use of signs, but not as an empirical law of "correct" thinking (Krämer, 1988). This seems to be not only in the nature of things as an epistemological principle, but also in some respects genetically determined: The brain seems to like working most effectively – thinking of higher neurofunctional "internal" rationalization effects, meaning less involved brain capacity. With our article, we can extend the discussion in mathematics education about the number sense (Dehaene, 1999) and the symbol sense (Arcavi, 2005) by an integrative cnA and thus open up an additional level of reasoning. For our brain, a task in a sign system (which is already well mastered, such as decimals, which are perceived as a total expression) seems to be easier to process than a new structure, which is perceived in a multipart structure (e.g., fractions). Thus, the following statement appears to be admissible for our brain: The further we deviate from a known sign system, the higher the cognitive effort to grasp it. This is exemplified by the following reproach, which Prediger (2009) attributes to a student: "You always make us think so much, can't we just calculate for once? That's less exhausting!"

For mathematics education, we must then ask whether formal calculus orientation or the development of semantic understanding should be in the foreground. Due to the processing in our brain, problem-solving contexts, for example, appear to be more cognitively demanding and challenging than a purely formal (unbound) calculus context. Prediger (2009) identifies the principle of *content-based thinking before calculation* as characteristic. It is characterized by the development of sustainable and diverse basic ideas through suitable sample situations and representations as the basis of content-based thinking, dwelling on the content until the learners themselves feel the need for shortcuts that relieve the burden of thinking by means of calculation, and consistent retention of content-based thinking even after the introduction of calculus, and until classwork.

A tentative inference from the mentioned studies might be that we should enrich as much calculus as we need with as much content-related thinking as we can. Working efficiently in sign systems seems appealing and important not only from an integrative cnA, but also to ensure that these skills do not remain in isolated domains: a connection to the world, to applications, to content references should always be sought (even if this is cognitively demanding). Only then can the effective calculus systems that the brain follows so willingly be useful for problem solving in mathematics classes, everyday life and work. Also, mathematical knowledge can be acquired and retained more sustainably if it is based on understanding (Prediger, 2009). As a caveat, it must be said that, according to our theory-based discussion, only a very few studies could be found to which we refer. These studies were conducive to our research question, and they use fMRI to support their claims. Our analysis in terms of our research question, the unloading processes of our brain through symbolic representations, should serve as a scientific impetus for further (also fMRI reviewed) studies in the future.

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